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Math 645
HW3

1. Find the points of intersection of the graphs of $y = x^2$ *and* $y = \dfrac{1}{2+x^2}$.

$$x^2 = \frac{1}{2+x^2}$$

$$\left(2+x^2\right)x^2 = 1$$

Substituting we have $x^4 + 2x^2 - 1 = 0$

$$x^2 = \frac{-2 \pm \sqrt{4-4(-1)}}{2} = \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2} = y$$

$$x = \pm\sqrt{-1 \pm \sqrt{2}}.$$

Checking all possible answers, the points of intersection are:
$\left(\sqrt{-1+\sqrt{2}}, -1+\sqrt{2}\right)$, $\left(\sqrt{-1-\sqrt{2}}, -1-\sqrt{2}\right)$.

2. Prove the the Chebyshev polynomials are orthogonal on [-1,1] with respect to the weight $(1-x^2)^{\frac{-1}{2}}$. That is, $\int_{-1}^{1} \frac{T_i(x)T_j(x)}{\sqrt{1-x^2}}dx = \begin{cases} 0, \textit{if } i \neq j \\ \frac{\pi}{2}, \textit{if } i = j \neq 0 \\ \pi, \textit{if } i = j = 0 \end{cases}$

Where $T_n(x) = \cos(n\cos^{-1} x)$.

Let $x = \cos\theta$. $T_n(\cos\theta) = \cos(n\cos^{-1}\left(\cos\theta\right)) = \cos(n\theta)$. Then we have for the case $i = j = 0$:

$$\int_{\pi}^{2\pi} \frac{T_0(\cos\theta)T_0(\cos\theta)}{\sqrt{1-\cos^2\theta}}dx = \int_{\pi}^{2\pi} \frac{\cos(n\cos^{-1}(\cos\theta))\cos(n\cos^{-1}(\cos\theta))}{\sqrt{1-\cos^2\theta}}dx$$

$$= \int_{\pi}^{2\pi} \frac{\cos(n\theta)\cos(n\theta)}{\sqrt{1-\cos^2\theta}}dx = \int_{\pi}^{2\pi} \frac{\cos^2(n\theta)}{\sqrt{1-\cos^2\theta}}dx = -\frac{\cos(n\theta)}{2}\sqrt{1-\cos^2\theta} + \frac{1}{2}\sin^{-1}\cos\theta \Big|_{\pi}^{2\pi}$$

$$= -\frac{\cos(n\theta)}{2}\sin^2\theta + \frac{1}{2}\sin^{-1}\left(\cos\theta\right)\Big|_{\pi}^{2\pi} = \left(-\frac{1}{2}(0) + \frac{1}{2}\sin^{-1}(1)\right) - \left(\frac{1}{2}(0) + \frac{1}{2}\sin^{-1}(-1)\right)$$

$$= \frac{\pi}{2} - -\frac{\pi}{2} = \pi$$

For the case $i = j \neq 0$ we have:

$$\int_{\pi}^{2\pi} \frac{T_j(\cos\theta)T_j(\cos\theta)}{\sqrt{1-\cos^2\theta}}dx = \int_{\pi}^{2\pi} \frac{\left[j\cos^j(\cos^{-1}(\cos\theta)) - (j-1)\right]\left[j\cos^j(\cos^{-1}(\cos\theta)) - (j-1)\right]}{\sqrt{1-\cos^2\theta}}dx$$

$$= \int_{\pi}^{2\pi} \frac{\left[j\cos^j(\theta) - (j-1)\right]^2}{\sqrt{1-\cos^2\theta}}dx = \int_{\pi}^{2\pi} \frac{\left[j^2\cos^{2j}(\theta) - 2j\cos^j(\theta)(j-1) + (j-1)^2\right]}{\sin^2\theta}dx = -\frac{\cos(n\theta)}{2}\sin^2\theta + \frac{1}{2}\sin$$

$$= \frac{\pi}{2}$$

For the case $i \neq j$ we have:

$$\int_{\pi}^{2\pi} \frac{T_i(\cos\theta)T_j(\cos\theta)}{\sqrt{1-\cos^2\theta}}dx = \int_{\pi}^{2\pi} \frac{\cos(i\cos^{-1}(\cos\theta))\cos(j\cos^{-1}(\cos\theta))}{\sqrt{1-\cos^2\theta}}dx$$

$$= \int_{\pi}^{2\pi} \frac{\cos(i\theta)\cos(j\theta)}{\sqrt{1-\cos^2\theta}}dx = -\frac{\cos(n\theta)}{2}\sqrt{1-\cos^2\theta} + \frac{1}{2}\sin^{-1}\cos\theta \Big|_{\pi}^{2\pi}$$

$$= -\frac{\cos(n\theta)}{2}\sin^2\theta + \frac{1}{2}\sin^{-1}\left(\cos\theta\right)\Big|_{\pi}^{2\pi} = \left(-\frac{1}{2}(1) + \frac{1}{2}\sin^{-1}(0)\right) - \left(\frac{1}{2}(-1) + \frac{1}{2}\sin^{-1}(0)\right)$$

$$= -\frac{1}{2} + \frac{1}{2} = 0$$

3. Let p be a polynomial for which $p(x^2+1) = x^4 + 5x^2 + 3$ for all x. Find $p(x)$.

Let $y = x^2 + 1$. *Then* $x = \sqrt{y-1}$. We will use this to find *p*(*x*).

$$p(x) = p(\sqrt{y-1}) = (\sqrt{y-1})^4 + 5(\sqrt{y-1})^2 + 3$$

$$= (y-1)^2 + 5(y-1) + 3$$
$$= y^2 - 2y + 1 + 5y - 5 + 3$$
$$= y^2 + 3y - 1$$

$So,\ p(x) = x^2 + 3x - 1$

4. Let a, b, and c be the three roots of $x^3 - 48x - 12$. What is the value of $a^3 + b^3 + c^3$?

Since a, b, and c are the roots, we have the following equations:

$$a^3 - 48a - 12 = 0$$
$$b^3 - 48b - 12 = 0$$
$$c^3 - 48c - 12 = 0$$

So, $a^3 + b^3 + c^3 = 48a + 48b + 48c + 36$

We also know that since a, b, and c are roots,

$$(x-a)(x-b)(x-c) = x^3 - 48x - 12$$
$$(x^2 - bx - ax + ab)(x-c) = x^3 - 48x - 12$$
$$x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc = x^3 - 48x - 12$$

Equating the coefficients, in particular of the x^2 term,

$$-(a+b+c) = 0,$$
$$so \quad a = -(b+c)$$
$$and \quad b+c = -a$$

Remember,

$$a^3 + b^3 + c^3 = 48a + 48b + 48c + 36 = 48(a+b+c) + 36$$

$$= 48(a-a) + 36 = 36.$$

So, $a^3 + b^3 + c^3 = 36$.

5. Let a and b be roots of $p(x) = x^3 + x - 1$. Prove that ab is a root of $q(x) = x^3 - x^2 - 1$.

Since a and b are roots we have

$a^3 + a - 1 = 0$

$b^3 + b - 1 = 0$

Since p(x) is a third degree polynomial we can call the third root c. Then we have

$(x-a)(x-b)(x-c) = x^3 + x - 1$

$(x^2 - ax - bx + ab)(x-c) = x^3 + x - 1$

$x^3 - cx^2 - ax^2 - bx^2 + acx + bcx + abx - abc = x^3 + x - 1$

$x^3 + (-a-b-c)x^2 + (ab+ac+bc)x - abc = x^3 + x - 1$

Equating the coefficients, in particular the constant term:

$abc = 1, \quad so\ c = \dfrac{1}{ab}$. We will use this fact to show that ab is a root of

$q(x) = x^3 - x^2 - 1$.

Now we have

$p(c) = p\left(\dfrac{1}{ab}\right) = \left(\dfrac{1}{ab}\right)^3 + \left(\dfrac{1}{ab}\right) - 1 = 0$

$= \dfrac{1}{a^3b^3} + \dfrac{1}{ab} - 1 = 0$

$= 1 + a^2b^2 - a^3b^3 = 0$

Multiplying both sides by -1 we have $a^3b^3 - a^2b^2 - 1 = (ab)^3 - (ab)^2 - 1 = 0$.

This is the same as $q(ab) = (ab)^3 - (ab)^2 - 1$, which equals 0, and so ab is a root of $q(x) = x^3 - x^2 - 1$.

6. Let $p(x)$ be a polynomial of degree four such that $p(2) = p(-2) = p(-3) = -1$ and $p(1) = p(-1) = 1$. Find $p(0)$.

$p(x) = ax^4 + bx^3 + cx^2 + dx + e$

$p(2) = 16a + 8b + 4c + 2d + e = -1$

$p(-2) = 16a - 8b + 4c - 2d + e = -1$

$p(-3) = 81a - 27b + 9c - 3d + e = -1$

$p(1) = a + b + c + d + e = 1$

$p(-1) = a - b + c - d + e = 1$

Using matrices to solve, we have $\begin{vmatrix} 16 & 8 & 4 & 2 & 1 \\ 16 & -8 & 4 & -2 & 1 \\ 81 & -27 & 9 & -3 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \end{vmatrix} \cdot \begin{vmatrix} a \\ b \\ c \\ d \\ e \end{vmatrix} = \begin{vmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 1 \end{vmatrix}$ corresponding to the form Ax = b

Solving, $\begin{vmatrix} a \\ b \\ c \\ d \\ e \end{vmatrix} = A^{-1} \begin{vmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 1 \end{vmatrix} = \begin{vmatrix} \frac{37}{444} \\ 0 \\ \frac{-481}{444} \\ 0 \\ 2 \end{vmatrix}$

So, $p(x) = \frac{37}{444}x^4 - \frac{481}{444}x^2 + 2$

$p(0) = 2$

7. Let $p(x) = 2 + 4x + 3x^2 + 5x^3 + 3x^4 + 4x^5 + 2x^6$ and define

$I_\alpha = \int_0^\infty \frac{x^\alpha}{p(x)} dx$ for $\alpha \in (-1, 5)$. Prove that I_α is smallest when $\alpha = 2$.

We know that $I_\alpha' = \frac{x^\alpha}{p(x)} = 0$ at maxima and minima, and

$I_\alpha'' = \frac{d}{dx}\frac{x^\alpha}{p(x)} = \frac{\alpha x^{\alpha-1} p(x) - x^\alpha p'(x)}{p^2(x)}$

Furthermore, we know that the second derivative is positive when $\alpha = 2$.

Therefore, $I_\alpha = \int_0^\infty \frac{x^\alpha}{p(x)} dx$ is smallest when $\alpha = 2$.